Chapter 1

Logic and Set Theory

1.1 Propositions

A **proposition** is a sentence that has a truth value: true (T) or false (F). It cannot be both true and false simultaneously. Some propositions have easily-determined truth values, such as "$\pi < 3$" and "It is 5:24pm". Other propositions may have truth values that we will never know, such as "Euler had eggs for breakfast on his 10th birthday".

Of course, not all sentences are propositions. For example, "$x^2 + 1 = 0$" could be true or false depending on which value is assigned to the variable x. Can you come up with a sentence that is neither true nor false?

There are a few very common ways to construct new propositions from given propositions P and Q.

- The **negation** of P (denoted by $\sim P$) is the proposition "not P".
- The **conjunction** of P and Q (denoted $P \wedge Q$) is the proposition "P and Q".
- The **disjunction** of P and Q (denoted $P \vee Q$) is the proposition "P or Q".

Note that P and $\sim P$ always have opposite truth values, $P \wedge Q$ is true precisely when both P and Q are true, and $P \vee Q$ is true precisely when at least one of P or Q is true. As an example, let A be the proposition "π is a rational number" and let B be the proposition "Lansing is the capital of Michigan". Then $A \wedge B$ is false, but $A \vee B$ is true. Can you determine the truth value of the proposition $\sim A \vee B$?

In general, the *form* of a proposition does not have a truth value until we have specific propositions to "plug in". We don't know, for example, the truth value of $P \wedge Q$ is true until we know the truth values of the propositions P and Q. This is similar to saying that $x^2 = 25$ does not have a truth value until we choose a specific value for x. For this reason, it is convenient to make charts that give us all possible truth values of a proposition, based on the truth values of the propositions it is made up of. We call these tables **truth tables**. Here are the truth tables for the negation, conjunction, and disjunction:

P	$\sim P$
T	F
F	T

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Let's now add in a third proposition R, and construct the truth table for the propositional form $(P \vee Q) \wedge \sim R$. We start with three columns, representing every possible combination of truth values for P, Q, and R (of which there are eight!). We then construct our desired propositional form column-by-column, providing truth values in each column along the way.

P	Q	R	$P \vee Q$	$\sim R$	$(P \vee Q) \wedge \sim R$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	F
F	F	F	F	T	F

The reader should construct the truth table for the propositional form $(P \vee Q) \vee (\sim P \wedge \sim Q)$ in order to see that this propositional form is true no matter what the truth values of P and Q are. In other words, the last column of your truth table should be all "T's". Any propositional form that has this property is called a **tautology**. On the other hand, and propositional form that is always false for every assignment of truth values to its components is called a **contradiction**. Can you come up with an example of a propositional form that is a contradiction? (Hint: The negation of a tautology is always a contradiction).

After the example above with P, Q, and R, one can easily imagine that propositional forms can become somewhat unwieldy. It is often useful, when possible, to write a propositional form in a simpler but equivalent way. We define two propositional forms to be **equivalent** if and only if they have the same truth tables. As an easy example, note that P and $\sim (\sim P)$ are equivalent propositional forms. (Provide the truth tables to show this). Another famous example of an equivalence of this sort is given by DeMorgan's Laws:

- $\sim (P \vee Q)$ is equivalent to $\sim P \wedge \sim Q$
- $\sim (P \wedge Q)$ is equivalent to $\sim P \vee \sim Q$

Again, the reader should provide the truth tables to verify these equivalences.

1.2 (Bi)conditionals

Consider the sentence "If you score 100% on the final exam, then you earn an A for the course". This "if-then" sentence is a proposition which has component propositions "you score 100% on the final exam" and "you earn an A for the course". The truth of the given "if-then" proposition depends on the truth values of the component propositions. The reader should take a moment to consider all of the cases possible here, of which there are four (each component can be either true or false). For example, if you score 100% on the final exam and you get an A for the course on your transcript, it should make intuitive sense that the "if-then" proposition above is true. That is, when both of the component propositions are true, then the "if-then" proposition is true. What about the other three cases? Before we use a truth table to show what happens in these cases, let us introduce some terminology. For propositions P and Q, the **conditional sentence** $P \Rightarrow Q$ is the proposition "If P, then Q." The name *conditional* refers to the fact that proposition P is giving conditions. The proposition P is called the **antecedent** and Q is called the **consequent**. The truth values of $P \Rightarrow Q$ are given in the following truth table.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Notice that $P \Rightarrow Q$ is true precisely when P is false or Q is true. In particular, one can think of the second line of the table above to mean that a promise was broken. For example, if you score 100% on the final exam (so the antecedent in the example above is true) but do not earn an A for the course (so the consequent in the example above is false) then it appears that what was promised to you did not happen. Otherwise, if you don't score 100% on the final exam, then the promise is not broken regardless of the grade you earn in the course. Furthermore, as mentioned above, if you do score 100% on the final exam and you earn an A in the course, then the promise was certainly not broken. It is very interesting to note here that *the proposition $P \Rightarrow Q$ is true whenever the antecedent is false!* So, for example, if one begins their "if-then" statement with "If the moon is made of green cheese" then regardless of the consequent, the conditional sentence is true. It is also quite interesting to note that the conditional $P \Rightarrow Q$ has a truth value regardless of any existing relationship between P and Q. The proposition "If $2 + 2 = 5$, then all humans are over 7 feet tall" is true. Similarly, the proposition "If 2+2=5, then some humans are over 7 feet tall" is also true.

You may be wondering at this point how we deal with familiar propositions from mathematics, like the following transitivity statement:

"If $2 < x < y$, then $2 < y$."

It seems impossible to assign a truth value to this proposition without choosing specific values for x and y. It is the case, however, the the conditional sentence above is true for every possible choice of x and y (convince yourself of this!) so that we are safe to say that this proposition is true. In general, it is important to note that when we say $P \Rightarrow Q$ is true, we are not claiming that either P or Q is true.

We have special names for particular conditional sentences, based on a given conditional sentence. The propositions $Q \Rightarrow P$, $\sim Q \Rightarrow \sim P$, and $\sim P \Rightarrow \sim Q$ are called the **converse**, **contrapositive**, and **inverse** of the proposition $P \Rightarrow Q$, respectively. It may be interesting to note that *"the contrapositive is the inverse of the converse"*. The reader should take a moment to give the converse, inverse, and contrapositive of the proposition "If $2 + 2 = 5$, then all humans are over 7 feet tall.", along with the truth values of each. In general, a conditional sentence is equivalent to its contrapositive, as is proved by examining the fifth and sixth columns of the following truth table, which are identical.

P	Q	$\sim P$	$\sim Q$	$P \Rightarrow Q$	$\sim Q \Rightarrow \sim P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The reader should easily be able to add two more columns to the table above, one for the inverse and one for the converse. These columns will show that, in general, a conditional sentence is neither equivalent to its inverse nor its converse.

We often need to examine a conditional sentence and its converse at the same time. That it, we want to examine the propositional form

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

To do so, for propositions P and Q, we introduce the **biconditional sentence** $P \Leftrightarrow Q$. $P \Leftrightarrow Q$ is the proposition "P if and only if Q.", and is true precisely when P and Q have the same truth values. (A very common abbreviation for the phrase "if and only if" is the triplet "iff", which we will use throughout the rest of this text.) Here is the corresponding truth table:

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

In the exercises at the end of the chapter, we will ask you to prove that $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$, along with a number of other equivalences. For now, we provide a glossary of some possible interpretations of the symbolic forms of conditionals and biconditionals.

$P \Rightarrow Q$ may be read as	$P \Leftrightarrow Q$ may be read as
If P, then Q	P if and only if Q
P implies Q	P is equivalent to Q
P is sufficient for Q	P is necessary and sufficient for Q
Q is necessary for P	
P only if Q	
Q whenever P	

As a final note for this section, we'll address a sort of "order of operations" for the connectives that we've introduced:

$$\sim, \wedge, \vee, \Rightarrow, \Leftrightarrow$$

are always applied in the order listed. We can regard this as saying that, in the order listed above, the connectives are applied to the smallest proposition possible. So, for example,

$$P \Leftrightarrow \sim Q \wedge R \Rightarrow S \text{ is the same as } P \Leftrightarrow (((\sim Q) \wedge R) \Rightarrow S)$$

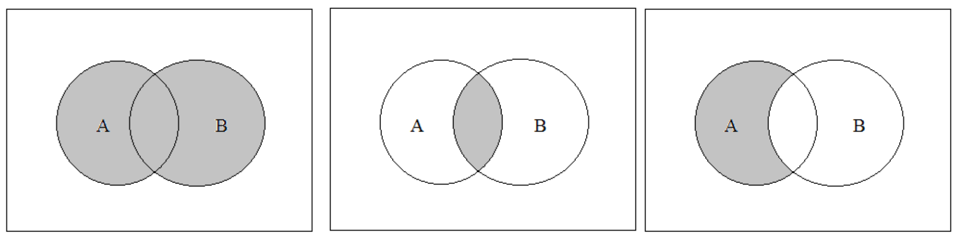
1.3 Operations With Sets

Just like we can combine numbers with operations like addition and multiplication, we can also combine sets in some special ways. Three of these operations with sets stand out: **union**, **intersection**, and **difference**. These are *binary* operations on sets because they take two sets and combine them into one. Let A and B be sets. Then:

- The **union** of A and B is denoted $A \cup B$ and is defined to be the set of all elements that are in A *or* B.

- The **intersection** of A and B is denoted $A \cap B$ and is defined to be the set of all elements that are in A *and* B.

- The **difference** of A and B is denoted $A - B$ and is defined to be the set of all elements that are in A but not in B.



A ∪ B   A ∩ B   A - B

These operations are pictured above using Venn Diagrams. For example, if $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{1, 3, 5, 7, 9, 11\}$, then

- $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 9, 11\}$

- $A \cap B = \{1, 3, 5, 7\}$

- $A - B = \{2, 4, 6\}$

The reader should experiment with these operations when the sets involved are intervals of real numbers. Notice that it may not be the case that two sets overlap. For example, the intersection $(1, 3] \cap (4, 5) = \varnothing$. In the event that the intersection of two sets is the empty set, we say that the sets are **disjoint**.

There is one more set operation that we will make frequent use of, called the **Cartesian product**, and denoted by $\times$. For two sets A and B, we define their cartesian product to be

$$A \times B = \{(a, b) \ : \ a \in A \text{ and } b \in B\}.$$

That is, the set of all *ordered* pairs with first element from A and second element from B. The reader should compute $A \times B$ for the sets A and B given above. The reader should also take a moment to visualize and describe $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, which we will also denote as $\mathbb{R}^3$.